AN IMPROVED STABILITY METHOD FOR LINEAR SYSTEMS WITH FAST-VARYING DELAYS

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Abstract: Stability of linear systems with uncertain bounded time-varying delays is studied under assumption that the nominal delay values are not equal to zero. An input-output approach to stability of such systems is known to be based on the bound of the L_2-norm of a certain integral operator. There exists a bound on this operator in two cases: in the case where the delay derivative is not greater than 1 and in the case without any constraints on the delay derivative. In the present note we fill the gap between the two cases by deriving a tight operator bound which is an increasing and continuous function of the delay derivative upper bound $d \geq 1$. For $d \to \infty$ the new bound corresponds to the second case and improves the existing bound. As a result, delay-derivative-dependent frequency-domain and time-domain stability criteria are derived for systems with the delay derivative greater than 1.

Keywords: time-varying delay, stability, input-output approach, L_2-norm.

1. INTRODUCTION

Two main approaches have been applied to stability analysis of linear systems with uncertain time-varying delay: a direct Lyapunov approach and an input-output approach (see e.g. Gu *et al.*, 2003), which reduces the stability analysis of the uncertain system to the analysis of the class of systems with the same nominal part but with additional inputs and outputs. In the existing literature the uncertain time-varying delay has been divided into two types: the *slowly-varying* delay (with delay derivative less than $d < 1$) and the *fast-varying delay* (without any constraints on the delay derivative) (see e.g. Kolmanovskii & Myshkis , 1999; Niculescu , 2001). Recently a third type of *moderately varying delay* has been revealed in Fridman & Shaked (2005), where the delay derivative is not greater than 1 (almost for all t). This has been obtained by applying the input-output approach to stability. It is known Gu *et al.* (2003), Kao & Lincoln (2004) that the latter approach to systems with time-varying bounded delays is based on the bound of the L_2-norm of a certain integral operator.

In the present paper we fill the gap between the case of the delay derivative not greater than 1 and the fast-varying delay by deriving a new integral operator bound. This bound is an increasing and continuous function of the delay derivative bound $d \geq 1$. In the limit case (where $d \to \infty$) which corresponds to the fast-varying delay, the new bound improves the existing one. As a result, improved frequency-domain and time-domain sta-

bility criteria are derived for systems with the delay derivative greater than 1.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. L_2 is the space of square integrable functions $v:[0,\infty)\to C^n$ with the norm $\|v\|_{L_2}=[\int_0^\infty \|v(t)\|^2dt]^{1/2}$, $\|A\|$ denotes the Euclidean norm of a $n\times n$ (real or complex) matrix A, which is equal to the maximum singular value of A. For a transfer function matrix of a stable system $G(s),\ s\in C$

$$\|G\|_\infty=\sup_{-\infty<w<\infty}\|G(iw)\|,\quad i=\sqrt{-1}.$$

2. PROBLEM FORMULATION

We consider the following linear system with uncertain time-varying delay $\tau(t)$:

$$\dot{x}(t)=A_0x(t)+A_1x(t-\tau(t)),\tag{1}$$

where $x(t)\in\mathcal{R}^n$ is the system state, $A_i,\ i=0,1$ are constant matrices.

The uncertain delay $\tau(t)$ has a form

$$\tau(t)=h+\eta(t),\qquad |\eta(t)|\le\mu\le h,\tag{2}$$

where h is a known nominal delay value and μ is a known upper bound on the delay uncertainty. In the existing literature Kolmanovskii & Myshkis (1999), Niculescu (2001), Gu *et al.* (2003) the following types of uncertain time-varying delays are usually considered:

Case A (slowly-varying delay): $\tau(t)$ is a differentiable almost everywhere function, satisfying

$$\dot{\tau}(t)=\dot{\eta}(t)\le d=1+p,\tag{3}$$

where $-1\le p<0$;

Case B (fast-varying delay): $\tau(t)$ is a measurable (e.g. piecewise-continuous) function.

Recently a *moderately-varying* delay with $\dot{\tau}(t)\le d=1$ was introduced in Fridman & Shaked (2005). In the present note we enlarge the latter class of delays as follows:

Case C (moderately-varying delay): $\tau(t)$ is a differentiable almost everywhere function, satisfying (3) with $p\ge 0$.

In the present note we will improve the stability results in cases B and C by applying input-output approach and by deriving new inequalities. The results are easily generalized to the case of any finite number of the delays.

We represent (1) in the form:

$$\dot{x}(t)=A_0x(t)+A_1x(t-h)-A_1\int_{-h-\eta}^{-h}\dot{x}(t+s)ds.\tag{4}$$

Following Fridman & Shaked (2005) we introduce the following auxiliary system:

$$\begin{aligned}&\dot{x}(t)=A_0x(t)+A_1x(t-h)+\mu A_1u(t),\\&y(t)=\sqrt{\mathcal{F}(p)}\dot{x}(t),\end{aligned}\tag{5}$$

with the feedback

$$u(t)=-\frac{1}{\mu\cdot\sqrt{\mathcal{F}(p)}}\int_{-h-\eta}^{-h}y(t+s)ds,\tag{6}$$

where $\mathcal{F}:[-1,\infty]\to R^+$ is a scalar function which will be shortly defined and p is given by (3). The results for the delay of case B correspond to $p=\infty$, i.e. to $\mathcal{F}(\infty)$ in the input-output model (5), (6). Substitution of (6) in (5) readily leads to (4).

We are looking for $\mathcal{F}(p)$ which satisfies the following inequality

$$||u||^2_{L_2}\le||y||^2_{L_2},\quad\forall\ y\in L_2[0,\infty),\ y\big|_{[-\infty,0]}\equiv 0,\tag{7}$$

where u is given by (6). This is equivalent to the fact that $\mu\sqrt{\mathcal{F}(p)}$ is an upper bound on the L_2-norm of the integral operator $\Delta:L_2[0,\infty)\to L_2[0,\infty)$

$$z(t)=\Delta y(t)=\int_{-h-\eta}^{-h}y(t+s)ds,\ y\big|_{[-\infty,0]}\equiv 0,\tag{8}$$

i.e. that

$$\begin{aligned}&\|z\|^2_{L_2}\le\mu^2\mathcal{F}(p)\|y\|^2_{L_2}\ ,\\&\forall y\in L_2[0,\infty),\ y\big|_{[-\infty,0]}\equiv 0.\end{aligned}\tag{9}$$

Our objective is to find $\mathcal{F}(p)$ (as small as possible) such that (7) (or equivalently (9)) holds.

For $-1\le p<0$ (case A) it was established in Gu *et al.* (2003) that $\mathcal{F}(p)$ can be chosen to be 1. For $p\ge 0$ the following was found in Fridman & Shaked (2005): $\mathcal{F}(0)=1$ and $\mathcal{F}(p)\equiv 2$ for $p\in(0,\infty]$.

We note that the value 1 of $\mathcal{F}(p)$ for $-1\le p\le 0$ can not be improved (i.e. chosen to be less than 1). Indeed, taking constant delay $\eta(t)\equiv\mu$, which satisfies the condition of case A for any $-1\le p\le 0$, we consider the functions $y_\theta(t)=1$ as $0\le t\le\theta$, and $y_\theta(t)=0$ as $t>\theta$. Using formula (6) with $\mathcal{F}(p)=1$ we immediately obtain

$$\|y_\theta\|^2_{L_2}=\theta^2,\quad\|u\|^2_{L_2}=(\theta-\mu)^2+\frac{2}{3}\mu^2\ ,$$

and hence $\|u\|_{L_2}/\|y_\theta\|_{L_2}\to 1$ as $\theta\to\infty$.

In the present paper we will improve the values of $\mathcal{F}(p)$ for $p>0$ by showing that $\mathcal{F}(p)$ can be chosen as a *continuous increasing function of* $p\ge 0$ satisfying $\mathcal{F}(0)=1$ (as in Fridman &

Shaked (2005)), but $\mathcal{F}(p) < \mathcal{F}(\infty) = 1.75$ for $p > 0$. The improved values of $\mathcal{F}(p)$ will readily lead to improved stability criteria.

3. MAIN RESULTS

3.1 New Bounds

Proofs of the Lemmas of this section are given in the Appendix.

Lemma 1. Consider case C. For all $y(t) \in L_2[0,\infty)$ and such that $y(t) = 0\ \forall t \le 0$ and for $u(t)$ given by (6) inequality (7) holds with $\mathcal{F}$ given by

$$\mathcal{F}(p) = \begin{cases} \dfrac{2p+1}{p+1}, & \text{if} \quad 0 \le p < 1, \\ \dfrac{7p-1}{4p}, & \text{if} \quad p \ge 1\ . \end{cases} \tag{10}$$

As it was mentioned above, $\mathcal{F}$ is increasing continuous function satisfying for $p > 0$ the following inequality: $1 = \mathcal{F}(0) < \mathcal{F}(p) < \lim_{p\to\infty} \mathcal{F}(p) = 7/4$.

Lemma 2. Consider case B. For all $y(t) \in L_2[0,\infty)$ and such that $y(t) = 0\ \forall t \le 0$ and for $u(t)$ given by (6) inequality (7) holds with $\mathcal{F}(\infty) := 7/4$.

Remark 1. The value $7/4 = 1.75$ for $\mathcal{F}(\infty)$ in Lemma 2 is not far from an optimal one. The following example shows that it cannot be less than 1.5. Namely, define scalar functions $y(t)$ and $\eta(t)$ by

$$y(t) = \begin{cases} t, & \text{if } 0 \le t \le \mu, \\ \mu - t, & \text{if } \mu \le t \le 2\mu, \\ 0, & \text{if } y(2\mu - y) < 0, \end{cases}$$

$$\eta(t) = \begin{cases} -\mu, & \text{if } t \le \mu, \\ \mu, & \text{if } t > \mu. \end{cases}$$

Setting in (6) $\mathcal{F}(\infty) = 3/2$ we have $u(t) = -\frac{1}{\mu\sqrt{3/2}} z(t)$, where

$$z(t+h) = \int_{t-\eta(t)}^{t} y(s)ds$$

$$= \begin{cases} -(t+\mu)^2/2, & \text{if } -\mu \le t+h \le 0, \\ -(\mu^2 + 2\mu t - 2t^2)/2, & \text{if } 0 < t+h \le \mu, \\ (6\mu t - 3\mu^2 - 2t^2)/2, & \text{if } \mu < t+h \le 2\mu, \\ (t-3\mu)^2/2, & \text{if } 2\mu < t+h \le 3\mu, \\ 0, & \text{otherwise.} \end{cases}$$

We achieve equality in (7) since

$$\|y\|^2_{L_2} = \frac{2}{3}\mu^3,\ \|u\|^2_{L_2} = \frac{2}{3\mu^2}\|z\|^2_{L_2} = \frac{2}{3\mu^2}\cdot\mu^5 = \frac{2}{3}\mu^3.$$

3.2 A Tight Frequency-Domain Stability Criterion

We assume

A1 Given the nominal value of the delay $h > 0$, the nominal system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \tag{11}$$

is asymptotically stable.

The auxiliary system (5) can be written as $y = Gu$ with the transfer matrix

$$G(s) = \sqrt{\mathcal{F}(p)} sI(sI - A_0 - A_1 e^{-hs})^{-1} \mu A_1. \tag{12}$$

By the small gain theorem (see e.g. Gu *et al.* (2003) the system (1) is input-output stable (and thus asymptotically stable, since the nominal system is time-invariant) if $\|G\|_\infty < 1$. A stronger result may be obtained by scaling G:

Theorem 1. Consider (1) with delay given by (2). Under A1 the system is asymptotically stable if there exists non-singular matrix X such that

$$\|G_X\|_\infty < 1, \quad G_X(s) = XG(s)X^{-1}, \tag{13}$$

where G is given by (12) with $\mathcal{F}(p)$ of (10) and where $p \in [0,\infty)$ corresponds to case C, while $\mathcal{F}(\infty) = 7/4$ corresponds to case B.

Remark 2. From Theorem 1 it follows that under A1 (1) is asymptotically stable if

$$\mu < \frac{1}{\sqrt{\mathcal{F}(p)}}\cdot k,\ k = \frac{1}{\|sI(sI - A_0 - A_1 e^{-hs})^{-1} A_1\|_\infty}.$$

By Fridman & Shaked (2005) $\mathcal{F}(p) = 2, p > 0$ and thus (1) (with $\dot{\tau}(t) \le 1+p, p > 0$ or with $\tau(t)$ of case B) is asymptotically stable for $\tau(t) \in [h-\mu, h+\mu]$, where $\mu < 0.7071k$. By the new bounds of Lemma 2 and Lemma 1 we obtain a wider stability intervals:

$$\begin{array}{llll} p = 0.1, & \dot{\tau}(t) \le 1.1, & \mathcal{F}(p) = 1.0909, & \mu < 0.9574k, \\ p = 0.5, & \dot{\tau}(t) \le 1.5, & \mathcal{F}(p) = 1.3333, & \mu < 0.8660k, \\ p = 1, & \dot{\tau}(t) \le 2, & \mathcal{F}(p) = 1.5, & \mu < 0.8165k, \\ p = \infty, & \text{case B}, & \mathcal{F}(p) = 1.75, & \mu < 0.7559k. \end{array} \tag{14}$$

3.3 On Improved Stability Criteria in the Time-Domain

By applying the time-domain results of Fridman & Shaked (2005) via descriptor model transformation with the corresponding simple Lyapunov-Krasovskii functional we obtain:

Theorem 2. System (1) is asymptotically stable for all delays of (2), if there exist $n \times n$ matrices

Table 1. Maximum value of μ

$d = 1$	$d = 1.1$	$d = 1.5$	$d = 2$	fast delay
0.384	0.367	0.331	0.313	0.289

$0 < P_1,\ P_2,\ P_3,\ S > 0, Y_1,\ Y_2,\ T, R, R_a$ such that the following Linear Matrix Inequality (LMI)

$$\left[\begin{array}{c|cc} & \mu P_2^T A_1 & 0 \\ \Gamma_n & \mu P_3^T A_1 & \mathcal{F}(p) R_a \\ & 0 & 0 \\ \hline * & -\mu R_a & 0 \\ * & * & -\mathcal{F}(p) R_a \end{array}\right] < 0, \tag{15}$$

where

$$\Gamma_n = \begin{bmatrix} \Psi_n & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - Y^T + \begin{bmatrix} T \\ 0 \end{bmatrix} & hY^T \\ * & -S - T - T^T & -hT \\ * & * & -hR \end{bmatrix},$$
$$\Psi_n = P^T \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^T \\ I & -I \end{bmatrix} P + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix}$$
$$+ \begin{bmatrix} Y \\ 0 \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix}^T,\ P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix},\ Y = [Y_1\ Y_2]. \tag{16}$$

is feasible. Here $\mathcal{F}(p)$ is given by (10) in case C and $\mathcal{F}(\infty) = 7/4$ in case B.

LMI (15) is convex in $\mathcal{F}(p)$ and thus the smaller values of $\mathcal{F}(p)$ lead to a less restrictive conditions. The time-domain criteria give sufficient conditions for the frequency domain Theorem 1.

Example (Kharitonov & Niculescu, 2003): Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - \tau(t)),$$
$$\tau(t) = 1 + \eta(t),\ |\eta(t)| \le \mu,\ \dot{\tau}(t) \le d. \tag{17}$$

In (Fridman, 2004) the maximum value of μ, for which the system is asymptotically stable, was found to be $\mu = 0.271$ for all $d \ge 1$. The latter result was less restrictive than the one by (Kharitonov & Niculescu, 2003). By the time domain criterion of (Fridman & Shaked, 2005) for $d = 1$ the corresponding value of μ is greater ($\mu = 0.384$), while for $d > 1$ the result is the same ($\mu = 0.271$). Theorem 2 of the present paper leads to a wider stability interval for $d > 1$ (see Table 1). Note that the results by Kao & Rantzer (2005) do not improve the existing results by descriptor approach and do not treat the case of $h - \mu > 0$.

4. CONCLUSIONS

Linear systems with bounded time-varying delays are analyzed under the assumption that the nominal delay values are not equal to zero. Two cases of delay are considered: case B (without any constraints on the delay derivative) and case C (where the delay derivative is not greater than $d \ge 1$). An input-output approach to stability of such systems is known to be based on the bound of the L_2-norm of a certain integral operator. In the present paper for the first time a tight $d-$ *dependent* bound is derived. The existing bound in case B is also improved. In the past, case C was treated as case B, which was restrictive. The new bounds lead to improved stability criteria and gives tools for further improvements.

5. APPENDIX

Proof of Lemma 2. Denote by $\varphi : \mathbb{R}^2 \to \{0, 1\}$ the characteristic function of the domain D in the positive quadrant, bounded by the line $s = t - h$ and by the graph of the function $s = t - h - \eta(t)$, i.e.,

$$\varphi(t, s) = \begin{cases} 1, & \text{if } (t - h - s)(t - h - \eta(t) - s) \le 0, \\ 0, & \text{if } (t - h - s)(t - h - \eta(t) - s) > 0 \end{cases}$$

(shaded region in Figure 1). Then $z(t)$ given by (8) satisfies the following:

$$\|z\|_{L_2}^2 = \int_0^\infty \left(\int_{-\infty}^\infty \varphi(t, s_1) y(s_1) ds_1 \right)^T$$
$$\times \left(\int_{-\infty}^\infty \varphi(t, s_2) y(s_2) ds_2 \right) dt$$
$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \left(\int_0^\infty \varphi(t, s_1) \varphi(t, s_2) dt \right)$$
$$\times y^T(s_1) y(s_2) ds_1 ds_2$$
$$= \int_{-\infty}^\infty \int_{-\infty}^\infty k(s_1, s_2) y^T(s_1) y(s_2) ds_1 ds_2,$$
$$k(s_1, s_2) = \int_0^\infty \varphi(t, s_1) \varphi(t, s_2) dt,\ s_1, s_2 \in \mathbb{R}\ .$$

Hence $\|z\|_{L_2}^2 \le \|\mathcal{K}\|_{L_2} \cdot \|y\|_{L_2}^2$, where $\|\mathcal{K}\|_{L_2}$ is the L_2-norm of the operator $\mathcal{K} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$

$$\mathcal{K}(f)(t) = \int_{-\infty}^\infty k(t, s) f(s) ds, \quad f \in L_2(\mathbb{R})\ .$$

By the Riesz-Thorin interpolation theorem (see, for example, Okikiolu (1971), Theorem 5.1.3), $\|\mathcal{K}\|_{L_2} \le \sqrt{\|\mathcal{K}\|_{L_1} \cdot \|\mathcal{K}\|_{L_\infty}}$. Since $k(s_1, s_2) \ge 0$ and $k(s_1, s_2) = k(s_2, s_1)$, by the well-known formulas for the L_1 and L_∞-norms, we have $\|\mathcal{K}\|_{L_1} = \|\mathcal{K}\|_{L_\infty} \le \sup_{s \in [0,\infty)} K(s)$, where $K(s) = \int_0^\infty k(s_1, s) ds_1$, and hence we decide that

$$\|\mathcal{K}\|_{L_2} \le \sup_{s \in [0,\infty)} K(s)\ . \tag{18}$$

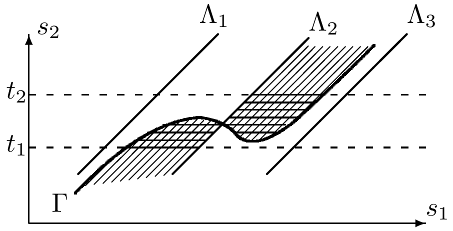
We shall show that $K(t) \le 7/4\ \mu^2$ for all $t \in [0, \infty)$.

Without loss of generality assume that $\eta(t) > 0$. Geometrically, $K(t)$ is the area of the part $D(t)$ of the domain D cut out by the strip $t - h \ge s_2 \ge t - h - \eta(t)$ (double shaded region in Figure 1).

Thus, $D(t)$ lies inside the parallelogram

$\Pi(t) := \{(s_1, s_2) \in \mathbb{R}^2 : t - h \le s_2 \le t - h - \eta(t),$
$s_1 - h - \mu \le s_2 \le s_1 - h + \mu\}$

(see Figure 2). Divide $\Pi(t)$ by the vertical lines $s_1 = t_4$ and $s_1 = t_5$ into two trapezes of total

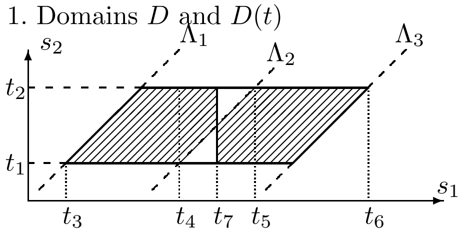


$t_1 = t - h - \eta(t), \quad t_2 = t - h$

$\Lambda_1 : s_2 = s_1 - h + \mu, \quad \Lambda_2 : s_2 = s_1 - h,$

$\Lambda_3 : s_2 = s_1 - h - \mu, \quad \Gamma : s_2 = s_1 - h - \eta(s_1)$

Fig. 1. Domains D and $D(t)$



$t_3 = t - \mu - \eta(t), \ t_4 = t - \eta(t), \ t_5 = t,$

$t_6 = t + \mu, \ t_7 = (t_4 + t_5)/2$

Fig. 2. Parallelogram $\Pi(t)$ and domain $D(t)$

area $\mu^2 - (\mu - \eta(t))^2$, and a square (see Figure 2). The intersection of $D(t)$ with a vertical line $s_1 = \sigma$, where $t_4 \le \sigma \le t_5$, is contained either in the segment $[t - h - \eta(t), \tau - h]$, or in the segment $[\tau - h, t - h]$, and hence the area of $D(t) \cap \{t - \eta(t) \le s_1 \le t\}$ does not exceed

$$\int_{t-\eta(t)}^{t} \max\{(\tau - (t - \eta(t)), t - \tau\} d\tau = \frac{3}{4}\eta(t)^2 \ .$$

So we derive the required bound (9) from the evident inequality

$$\mu^2 - (\mu - \eta(t))^2 + \frac{3}{4}\eta(t)^2 \le \frac{7}{4}\mu^2 \ ,$$

which geometrically means that the maximal area domain $D(t)$ looks as the shaded region in Figure 2.

Proof of Lemma 1. We interpret the function $\mathcal{F}(p)$ geometrically as follows:

- for $p \ge 1$, $\mathcal{F}(p)\mu^2$ is the area of the domain

$$D_p := \{(t, s) \in \mathbb{R}^2 \ : \ 0 \le s \le \mu, \ t - 2\mu \le s \le t, \ (t - \mu - s)(2s + 2pt - (3p + 1)\mu)) \ge 0\}$$

(shaded region in Figure 3(a)),

- for $0 \le p < 1$, $\mathcal{F}(p)\mu^2$ is the area of the domain

$$D_p = \{(t, s) \in \mathbb{R}^2 \ : \ 0 \le s \le \mu, \ t - 2\mu \le s \le t, \ (t - \mu - s)(s + pt - 2p\mu) \le 0\}$$

(shaded region in Figure 3(b)).

As in the proof of Lemma 2, we estimate from above the value of $K(s_2)$. i.e., the area of the domain $D(t)$.

Without loss of generality we assume that η is a smooth function, whose zero locus is locally

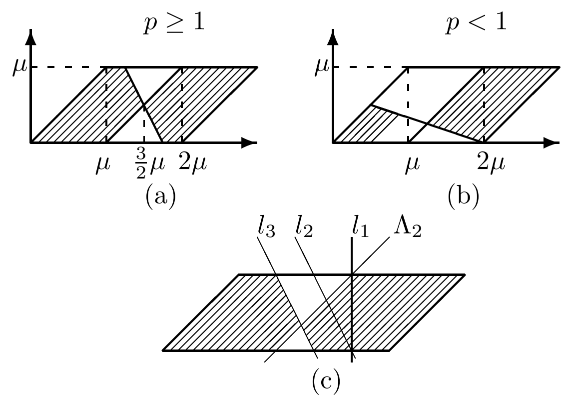


Fig. 3. The case $N = 1$

finite and consists of only simple roots. Fix $t > 0$. Assume that $\eta(t) > 0$, which, in particular, means that the line $s_1 = t$ crosses the domain $D(t)$ along the segment $t_1 \le s_2 \le t_2$ (cf. Figure 2). Put $N = \#(\eta^{-1}(0) \cap (t - \mu - \eta(t), t + \mu))$ (i.e., the number of zeroes in the interval (t_3, t_6) in Figure 2).

Consider few possibilities.

Step 1. Suppose that $N = 0$. Then $D(t)$ is contained in the part of the parallelogram $\Pi(t)$, right to the line $s_2 = s_1 - h$ (see Figure 2). Then its area does not exceed $\eta(t)\mu \le \mu^2 \le \mathcal{F}(p)\mu^2$.

Step 2. Suppose that $N = 1$. The zero $\tau = \eta^{-1}(0) \cap (t - \mu - \eta(t), t + \mu)$ corresponds to an intersection point of the graphs of the functions $s_2 = s_1 - h$ and $s_2 = s_1 - h - \eta(s_1)$. This intersection point lies either right to the line $l_1 := \{s_1 = t\}$ or below the line $l_2 := \{s_2 = -ps_1 + (p - 1)t - h - \eta(t)\}$ (see Figure 3(c)). In the former case, the domain $D(t)$ remains below the line $s_2 = s_1 - h$, that is we have the upper bound from Step 1. In the latter case, the part of domain $D(t)$ in the half-plane $s_1 \ge \tau$ should lie below the line $s_2 = s_1 - h$ and above the line $l_3 = \{s_2 = -ps_1 + (p - 1)\tau - h\}$, and the part of $D(t)$ in the half-plane $s_1 \le \tau$ should lie below the line l_3 and above the line $s_2 = s_1 - h$ (shaded region in Figure 3(c)). It is an elementary geometry exercise to show that the area of the shaded region in Figure 3(c) does not exceed $\mathcal{F}(p)\mu^2$.

Step 3. We intend to show that the case $N > 1$ reduces to the above considered cases $N = 0$ or 1. For, we need the following auxiliary geometric statement. Consider the parallelogram $\Pi(t)$, some points x_1, x_2, x_3 with decreasing coordinates, lying on the line $s_2 = s_1 - h$ below the line $l_2 = \{s_2 = -ps_1 + (p - 1)t - h - \eta(t)\}$ (see Figure 4(a)). Draw the lines l_4, l_5 with slope $-p$ through the points x_1, x_3, respectively, and the vertical line l_6 through x_2. Denote by $F(x_2)$ the area of the domain $\delta(x_2)$, lying inside $\Pi(t)$, between the lines l_4, l_5, and in the two sectors, bounded by the lines

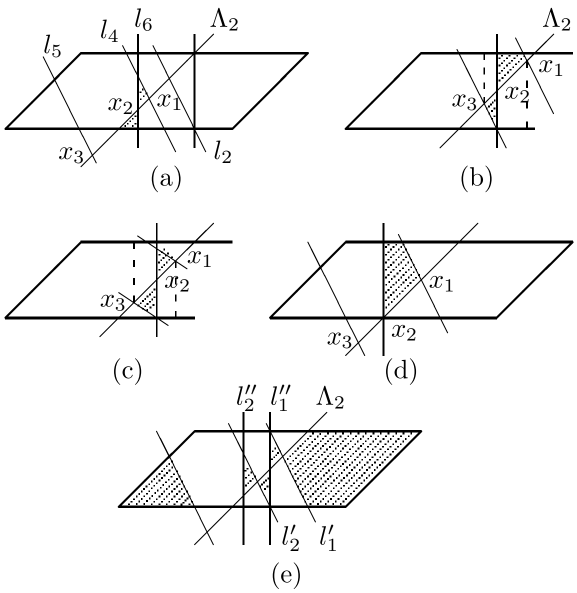


Fig. 4. The case $N > 1$

l_6 and $s_2 = s_1 - h$ as marked by dots in Figure 4(a). Consider $F(x_2)$ as a function of the point x_2 running along the segment $[x_1, x_3]$. We claim that F attains its maximum either at $x_2 = x_1$ or at $x_2 = x_3$.

Indeed, when x_2 runs along a subsegment of $[x_1, x_3]$ such that the combinatorial type of $\delta(x_2)$ does not change, F is a quadratic polynomial in the abscissa τ of x_2 with the positive second derivative. That is, in this subsegment, F attains its maximum at an endpoint. If such an endpoint differs from x_1 and x_3, then it corresponds to the situation when

- either x_2 belongs to a side of $\Pi(t)$ (see, for example, Figure 4(d));
- or the domain $\delta(x_2)$ intersects with some side of $\Pi(t)$ at a point (see, Figure 4(b,c)).

In the former situation, $\delta(x_2)$ entirely lies right to the vertical line through x_2, and then monotonically grows as x_2 tends to x_3. In the latter situation, when replacing x_2 by x_1, the domain $\delta(x_2)$ turns into the domain $\delta(x_1)$ by getting rid of $\delta(x_2) \cap \{s_1 \le \tau\}$ and adding the fragment δ_- (the trapeze, bounded by the vertical line through x_2, dashed line, the upper side of $\Pi(t)$, and the line $s_2 = s_1 - h$ in Figure 4(b)), and when replacing x_2 by x_3, the domain $\delta(x_2)$ turns into the domain $\delta(x_3)$ by by getting rid of $\delta(x_2) \cap \{s_1 \ge \tau\}$ and adding the fragment δ_+ (the trapeze, bounded by the vertical line through x_2, dashed line, the lower side of $\Pi(t)$, and the line $s_2 = s_1 - h$ in Figure 4(b)). One can easy see that the area of $\delta(x_2)$ in the above situation is less than the maximum of the areas of $\delta(x_1)$ and $\delta(x_3)$. The same conclusion one can derive in the situation, presented in Figure 4(c).

Step 4. Suppose that $N > 1$. We can take N to be odd, adding if necessary one more zero close to $t - \mu - \eta(t)$. Denote by $x_1, x_2, ..., x_n$ the corresponding intersection points of the graph of $s_2 = s_1 - h - \eta(s_1)$ with the line $s_2 = s_1 - h$, numbered by the decreasing coordinates. Through each point x_{2i-1} we draw a line l_i' with slope $-p$, $1 \le i \le (n+1)/2$, and through each point x_{2i} we draw a vertical line l_i'', $1 \le i \le n/2$. Thus, $D(t)$ is the union of the regions in $\Pi(t)$, bounded by the introduced lines as follows (marked by dots in Figure 4(e)):

- above the line l_1' and below the line $s_2 = s_1 - h$,
- below the line l_i', right to the line l_i'', and above the line $s_2 = s_1 - h$, $1 \le i \le n/2$,
- left to the line l_i'', above the line l_{i+1}' , and below the line $s_2 = s_1 - h$, $1 \le i \le n/2$,
- below the line $l_{(n+1)/2}'$ and above the line $s_2 = s_1 - h$.

Using the statement of Step 3, we can move the point x_2 either to the position x_1, or x_3 and increase the area of $\delta(t)$. On the other hand, each of these limit positions for x_2 means that we, in fact, have reduced two zeroes of η in the interval $(t - \mu - \eta(t), t + \mu)$. Thus, we inductively come to the case $N = 1$ treated in Step 2.

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